Johan's Seminar on (derived) Direct Summand Conj.

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DSC If A is a reg. local ring and $A \subseteq B$ is module finite extn of rings, Then $A \longrightarrow B$ splits as an A-mod. map

MC (monomial) Let A be a Noetherian local ring, let $x_1, \dots, x_d$ be a system of parameters (s.o.p.), then $\forall\, n \geq 0$ $(x_1, \dots, x_d)^n \not\subseteq (x_1^{n+1}, \dots, x_d^{n+1})$

Canonical elt conj. (CEC) (A, m, k) Noeth. local, $x_1, \dots, x_d$ s.o.p., $P_\cdot \longrightarrow k$ minimal free resolution. $\varphi : K_\cdot(\underline{x}) \longrightarrow P_\cdot$ lifting id_k. Then $\varphi_d : A = K_d(\underline{x}) \to P_d$ is not zero

$$\begin{array}{ccc} K_\cdot(\underline{x}) & \longrightarrow & P_\cdot \\ & \searrow \; k \; \swarrow & \end{array}$$

$0 \to Syz_d \longrightarrow P_{d-1} \longrightarrow \cdots \longrightarrow P_0 \to k \to 0$ gives $\mathcal{E}_A \in Ext_A^d(k, Syz_d)$

$$\begin{array}{ccc} \mathcal{E}_A & \in & Ext_A^d(k, Syz_d) \\ \downarrow & & \downarrow \\ \text{Canonical element} \longrightarrow \eta_A & \in & H_m^d(Syz_d) \end{array}$$

Lemma C.E.C $\Leftrightarrow$ $\eta_A \neq 0$ $\forall A$.

F A Noetherian, $A \subseteq B$ integral, M A-mod. If $M \otimes_A B$ is B-flat then M is A-flat.

Rmk. Interesting case: $M \otimes_A B = 0 \Rightarrow M = 0$.

Thm (Hochster, Ohi) $DSC \Leftrightarrow MC \Leftrightarrow CEC \Leftrightarrow F$.

Lemma DSC is true in char. 0.

pf. $A \subseteq B$ as in DSC w/ $\mathbb{Q} \subseteq A$. Replace B by $B/\mathfrak{q}$ where $\mathfrak{q} \subseteq B$ s.t. $(0) = \mathfrak{q} \cap A$. Then

$$\begin{array}{ccl} B & \subseteq & f.f.(B) = L \\ \uparrow & & \cup \quad \downarrow \tfrac{1}{[L:K]} tr_{L/K} \\ A & \subseteq & f.f.(A) = K \end{array}$$

Rmk For this pf to work it already suffices to assume A is normal domain.

Lemma DSC $\Rightarrow$ MC when $k \subseteq A$ for a field k.

pf. Given A and s.o.p. $x_1, \dots, x_d$, we get (after replacing A by $\hat{A}$)
$k[[x_1, \dots, x_d]] \hookrightarrow A$ finite. So it splits by DSC.
Therefore $(x_1^{n+1}A + \cdots + x_d^{n+1}A) \cap k[[x_1, \dots, x_d]] = (x_1^{n+1}, \dots, x_d^{n+1})$
So it suffices to check MC for rings of the form $k[[x_1, \dots, x_d]]$ ∎

Lemma MC $\Rightarrow$ DSC.

Idea using duality. $A \subseteq B$ as in DSC.

$(A \to B \text{ splits}) \Leftrightarrow (A \cong \omega_A \xleftarrow{\text{tr}} RHom_A(B, \omega_A) = \omega_B \text{ splits})$

$\overset{\text{local duality}}{\Longleftrightarrow} (R\Gamma_m(A) \longrightarrow R\Gamma_m(B) \text{ splits}) \overset{A \text{ regular}}{\Longleftrightarrow} E \cong H_m^d \to H_m^d(B)$ inj

Local duality $RHom(K^\bullet, \omega_A)^\wedge \cong RHom_A(R\Gamma_m(K^\bullet), E)$ where E is an inj. hull of k.

Lemma DSC in char. p

<u>Hochster</u> Prove M.C. $k[[x_1, \dots, x_n]] = R \hookrightarrow A$ finite.
choose a $\varphi: A \to R$ $\varphi(1) \neq 0$ and use Frob^n.

<u>Bhargav</u> Prove DSC in char. p directly.

Derived Direct Summand Conj. (DDSC) A reg. local, $f: X \to S = \mathrm{Spec}(A)$ proper & surj. Then $\mathcal{O}_S \longrightarrow Rf_*\mathcal{O}_X$ splits in $D(\mathcal{O}_S)$.

Defn S is a <u>derived splinter</u> if DDSC holds.

Thm (Kovacs) In char. 0 we have
S derived splinter $\Leftrightarrow$ S has only rat'l singularities!

Thm (Bhargav) In char p we have splinter $\Leftrightarrow$ derived splinter.

Almost direct Summand Conj Thm. $\mathbb{Z}_p[T_2, \cdots, T_d] \xrightarrow{\text{smooth}} A \overset{\text{finite}}{\subseteq} B$

If $A[\frac{1}{pT_2 \cdots T_d}] \subseteq B[\frac{1}{pT_2 \cdots T_d}]$ is étale, then $A \to B$ splits

Thm (André) If $A_0 \subseteq B_0$ finite extn of Noetherian rings and A_0 is regular, then it splits as an A_0-mod. map

Reduction We may assume A_0 is p-adically complete, p-torsion free and A_0/pA_0 is smooth over a perfect field k.

Elkik: A_0 = p-adic completion of a smooth $W(k) = W$-alg

Kedlaya: After localizing, we can pick $W[x_1, \cdots, x_d] \xrightarrow[\text{étale}]{\text{finite}} A_0$

Notation: $A_{\infty,0} = \widehat{A_0[p^{1/p^\infty}, x_i^{1/p^\infty}]}$ integral pftd K°-alg.

$K = \mathrm{Frac}(W[p^{1/p^\infty}]^\wedge)$ $K^\circ = W[p^{1/p^\infty}]^\wedge$

Defn A K°-alg. A is called int'l pftd if

(0) A is p-adically cplt, p torsion free.

(1) $f \in A[\frac{1}{p}]$, $p^\varepsilon f \in A$ $\forall \varepsilon = \frac{1}{p^n} > 0 \Rightarrow f \in A$.

(2) Frob: $A/(p^{1/p}) \longrightarrow A/(p)$ isom.

Fact Cat of int'l pftd K°-alg $\overset{\sim}{\Longleftrightarrow}$ Cat of pftd K-alg.

Now fix $g \in A_0$, nonzero-divisor in $A/(p)$ s.t.

$A_0[\frac{1}{gp}] \longrightarrow B_0[\frac{1}{pg}]$ étale

Defn $A_\infty = \widehat{\mathrm{colim}_l A_{\infty,l}}$ $A_{\infty,l} = A_{\infty,0}\langle T^{1/p^\infty}\rangle\langle \frac{T-g}{p^l}\rangle$.

Nb $T = g$ in A_∞!

Thm 2.3 (André) A_∞ almost isom to an integral pftd K°-alg and $A_{\infty,0} \longrightarrow A_\infty$ is almost faithfully flat w.r.t. p^{1/p^∞}.

pf of DSC: Consider $0 \to A_0 \to B_0 \to Q_0 \to 0$

$\alpha_0 \in \mathrm{Ext}^1_{A_0}(Q_0, A_0) = \mathrm{Hom}_{D(A_0)}(Q_0, A_0[1])$

"$\alpha = 0$" $\Leftrightarrow$ $(\alpha_0 \bmod p^m) \in \mathrm{Hom}_{D(A_0)}(Q_0, A_0/(p^m)[1])$ is 0 $\forall m$

Suppose not, $(\alpha_0 \bmod p^m) \neq 0$ for some $m \geq 3$

Then $I = \mathrm{Ann}_{A_0/(p^m)}(\alpha_0 \bmod p^m) \neq A_0/(p^m)$.

$\overset{\text{Krull's}}{\underset{\text{intersection Thm}}{\Longrightarrow}}$ $p^2 g \notin I^{p^k}$ for some $k \geq 0$.

Lemma: Because $A_{\infty,0}/pA_{\infty,0} \longrightarrow A_\infty/pA_\infty$ is almost f.f. we get $(pg)^{1/p^k} \notin \mathrm{Ann}(\alpha_\infty \bmod p^m)$, where

$\alpha_\infty \bmod p^m : Q_0 \otimes^{\mathbb{L}}_{A_0} A_\infty \longrightarrow A_\infty/(p^m)[1]$

Solution: Enough to show $\alpha_\infty \bmod p^m$ is almost 0 w.r.t. $(pg)^{1/p^\infty}$.

Look at $A_\infty\langle \frac{p^n}{g} \rangle$

Thm 4.2 (Bhargav) The map $\{A_\infty/(p^m)\}_{n\geq 1} \longrightarrow \{A_\infty\langle\frac{p^n}{g}\rangle/(p^m)\}_{n\geq 1}$ is an almost-pro-isom. w.r.t. $(pg)^{1/p^\infty}$.

observe that $A_\infty\langle\frac{p^n}{g}\rangle \longrightarrow B_0 \otimes^{\mathbb{L}}_{A_0} A_\infty\langle\frac{p^n}{g}\rangle$ $(\dashrightarrow C_n)$ finite étale after inverting p.

Almost purity (Schulze, pftd Spaces 7.9): One can find C_n as above w/ $A_\infty\langle\frac{p^n}{g}\rangle \longrightarrow C_n$ almost finite étale. Then we get image in $\mathrm{Hom}_{D(A_\infty)}(Q_0 \otimes^{\mathbb{L}}_{A_0} A_\infty, A_\infty\langle\frac{p^n}{g}\rangle/(p^m)[1])$ is 0, b/c we have $A_\infty\langle\frac{p^n}{g}\rangle \longrightarrow C_n$, $\exists$ almost section.

Almost Mathematics

Setup V is a valuation ring. $\mathfrak{m}$ max'l ideal w/ $\mathfrak{m}^2 = \mathfrak{m}$. ($\Rightarrow \mathfrak{m} \otimes \mathfrak{m} = \mathfrak{m}$)

Defn A V-mod. is almost zero if $\mathfrak{m} M = 0$.

Prop. almost zero V-mod form a Serre subset of V-mod's

Defn. V^a-mod is the cat. w/ obj V-mod., but
$f: M \longrightarrow N$ an isom. iff ker & coker $=^a 0$.

Prop. $\mathrm{Hom}_{V^a}(M^a, N^a) \cong \mathrm{Hom}_V(\mathfrak{m} \otimes M, N)$.

Prop. $V\text{-mod} \longrightarrow V^a\text{-mod}$ has left adjoint $(M^a)_! = \mathfrak{m} \otimes M$, right adjoint $(M^a)_* = \mathrm{Hom}_V(\mathfrak{m}, M)$.

Thm (Almost purity) If A is a pftd Tate ring & B/A fin. étale, then B is pftd & B° is almost finite étale $/A^\circ$.

<u>Note</u> V^a-mod has $\otimes$ & Hom.

Defn. If A is a V-alg. & M is an A-mod, then M is an <u>almost flat</u> A-mod if $\otimes M^a$ is exact on V^a-mod
$\Leftrightarrow \mathrm{Tor}_i^A(M,-)$ is A^a almost zero for $i \geq 1$.

Defn. Let $f: A \longrightarrow B$ be a map of V^a-alg's. f is <u>almost unramified</u> if $\exists\ e \in (B \otimes_A B)_*$ s.t. $e^2 = e$. $\mu: (B \otimes_A B)_* \longrightarrow B_*$ sends $\mu(e) = 1$
$e \cdot \ker(\mu) = 0$.
$(\mathrm{Hom}_V(\mathfrak{m}, A) \otimes_V \mathrm{Hom}_V(\mathfrak{m}, A) \longrightarrow \mathrm{Hom}_V(\mathfrak{m} \otimes_V \mathfrak{m}, A \otimes A) = \mathrm{Hom}_V(\mathfrak{m}, A \otimes A)$
$\longrightarrow \mathrm{Hom}_V(\mathfrak{m}, A))$.

Defn B°/A° almost fin. étale means it is almost flat, almost unramified and <u>almost f.g.</u>